Section 15.6:
Triple Integrals

What We’ll Learn In Section 15.6

1. Triple Integrals Over A Rectangular Box
2. Triple Integrals Over General Bounded Regions

Recall: Integrals of a 1-variable function over an interval

Let $f(x)$ be a continuous function on [a, b]. ($f(x)$ not necessarily ≥ 0)

Divide the interval [a, b] into n subintervals.

In each subinterval $[x_{i-1}, x_i]$, pick a point x_i^*.

$f(x)$ $f(x_i^*)\,\Delta x$

Calculate $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$

This sum approximates some quantity you're interested in

Then $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

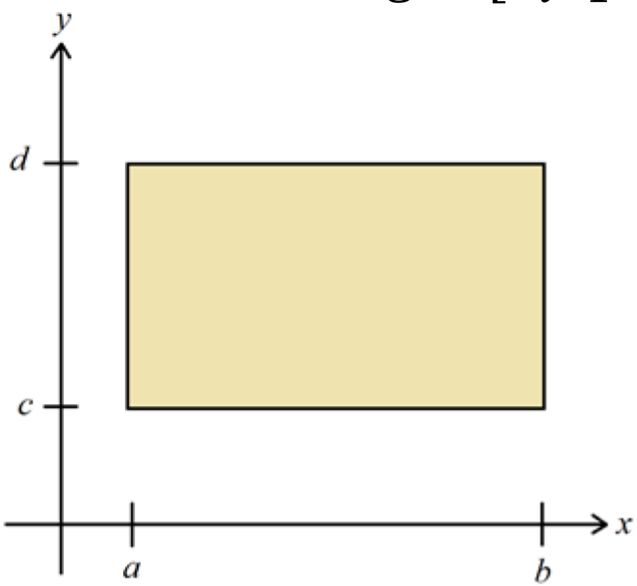
gives you the exact value of the quantity you're interested in

Recall: Integrals of a 2-variable function over a rectangle

Let $f(x, y)$ be a continuous function on $[a, b] \times [c, d]$.
($f(x, y)$ not necessarily ≥ 0)

Divide the interval [a, b] into m subintervals.
Divide the interval [c, d] into n subintervals.

In each sub-rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$, pick a point (x_{ij}^*, y_{ij}^*).



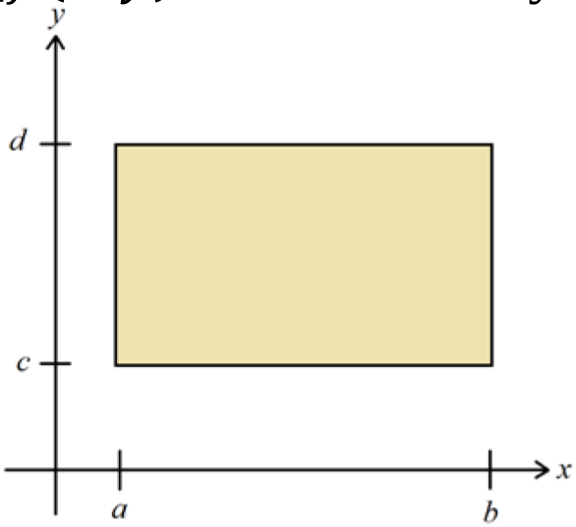
$f(x, y)$ $\quad$ $f(x_{ij}^*, y_{ij}^*)\,\Delta A$

Calculate $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$

This sum approximates some quantity you're interested in

Recall: Integrals of a 2-variable function over a rectangle

Let $f(x, y)$ be a continuous function on $[a, b] \times [c, d]$.
($f(x, y)$ not necessarily ≥ 0)



Calculate $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$

This sum approximates some quantity you're interested in

Then $\displaystyle\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$

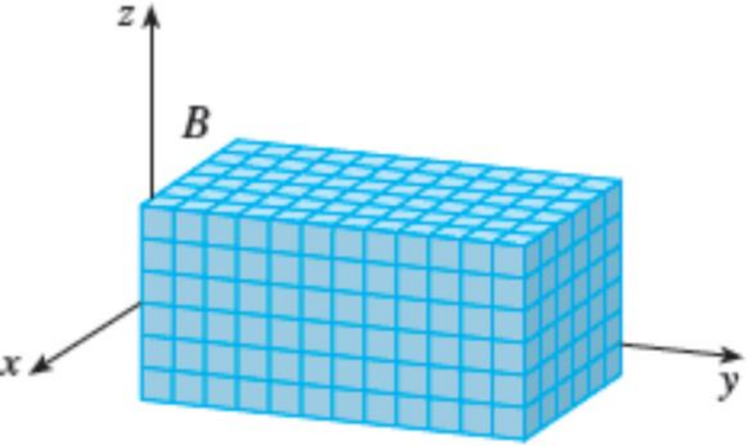
gives you the exact value of the quantity you're interested in

1. Triple Integrals Over A Rectangular Box

Let $f(x, y, z)$ be a continuous function on $[a, b] \times [c, d] \times [e, f]$.
($f(x, y, z)$ not necessarily ≥ 0)

Divide the interval [a, b] into l subintervals.
Divide the interval [c, d] into m subintervals.
Divide the interval [e, f] into n subintervals.

In each sub-rectangular box $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$, pick a point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$.

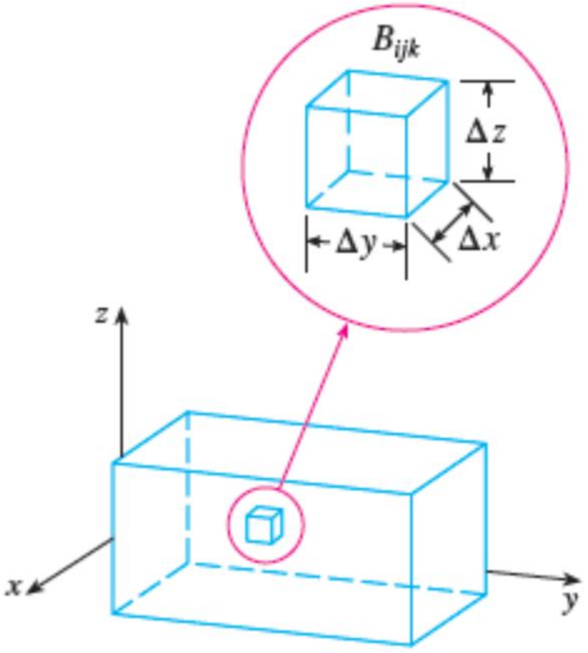


1. Triple Integrals Over A Rectangular Box

Let $f(x, y, z)$ be a continuous function on $[a, b] \times [c, d] \times [e, f]$.
($f(x, y, z)$ not necessarily ≥ 0)

Divide the interval [a, b] into *l* subintervals.
Divide the interval [c, d] into *m* subintervals.
Divide the interval [e, f] into *n* subintervals.

In each sub-rectangular box $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$, pick a point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$.



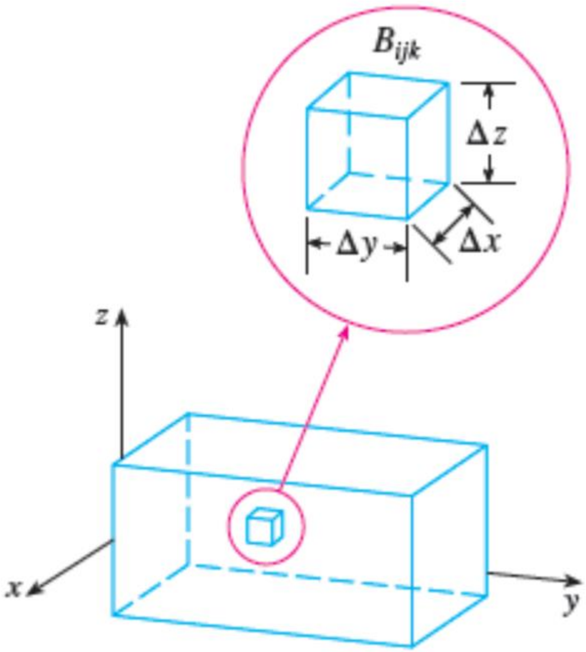
$f(x, y, z)$ $\qquad$ $f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V$

Calculate $\displaystyle\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V$

This sum approximates some quantity you're interested in

1. Triple Integrals Over A Rectangular Box

Let $f(x, y)$ be a continuous function on $[a, b] \times [c, d]$.
($f(x, y)$ not necessarily ≥ 0)



Calculate $$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V$$

This sum approximates some quantity you're interested in

Then $$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V$$

gives you the exact value of the quantity you're interested in

1. Triple Integrals Over A Rectangular Box

Ex 0: Use the Midpoint Rule for triple integrals to estimate the value of the following integral by dividing B into 8 sub-boxes of equal size.

$$\iiint_B \sqrt{x}e^{xyz}\,dV \qquad B = \{\,(x, y, z) \mid 0 \le x \le 4\,, 0 \le y \le 1\,, 0 \le z \le 2\,\}$$

1. Triple Integrals Over A Rectangular Box

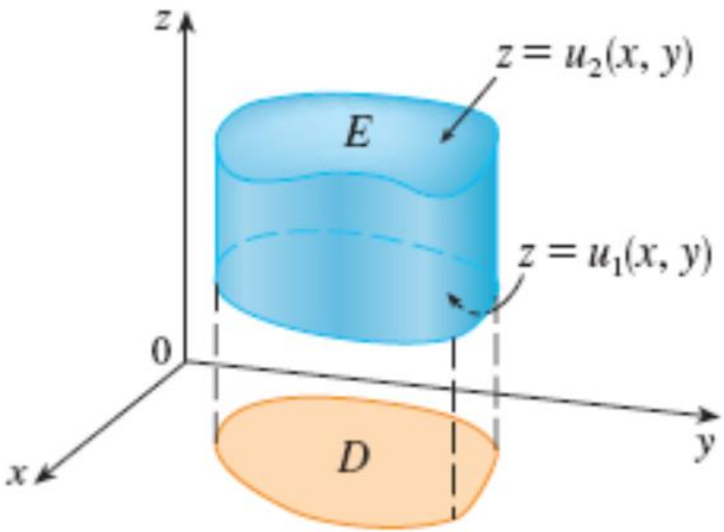
Ex 1: Evaluate the triple integral $\iiint_B xyz^2 \, dV$, where B is the rectangular box given by $B = \{ (x, y, z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3 \}$

2. Triple Integrals Over General Bounded Regions

<u>Type 1 Regions (3-dim):</u>

A solid region E is a <u>type 1 region</u> if it is all points in space "over" a plane region D in the xy-plane that are between the graphs of 2 surfaces (2-variable functions of x and y) $u_1(x,y)$ and $u_2(x,y)$.

$$E = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D \ and \ u_1(x,y) \le z \le u_2(x,y) \}$$



D will be a rectangle,
(2-dim) type 1 or type 2 region,
or a region we can easily describe
in polar coordinates

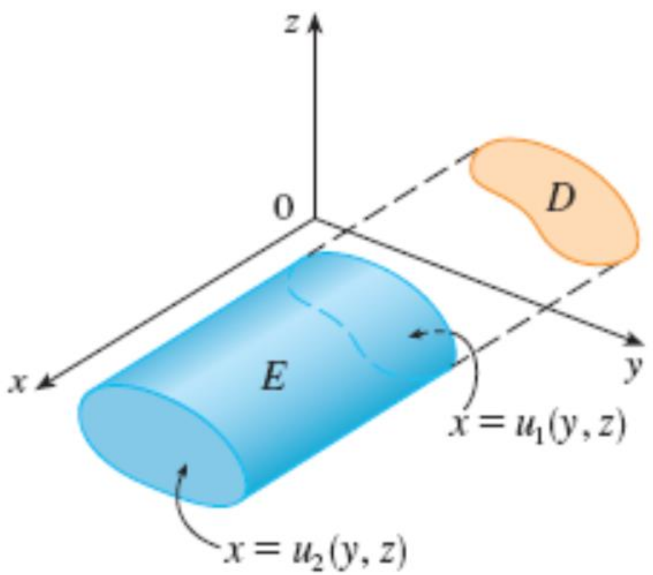
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

2. Triple Integrals Over General Bounded Regions

Type 2 Regions (3-dim):

A solid region E is a type 2 region if it is all points in space "in front" of a plane region D in the yz-plane that are between the graphs of 2 surfaces (2-variable functions of y and z) $u_1(y,z)$ and $u_2(y,z)$.

$$E = \{\, (x,y,z) \in \mathbb{R}^3 \;\middle|\; (y,z) \in D \;\; and \;\; u_1(y,z) \leq x \leq u_2(y,z) \,\}$$



D will be a rectangle,
(2-dim) type 1 or type 2 region,
or a region we can easily describe
in polar coordinates

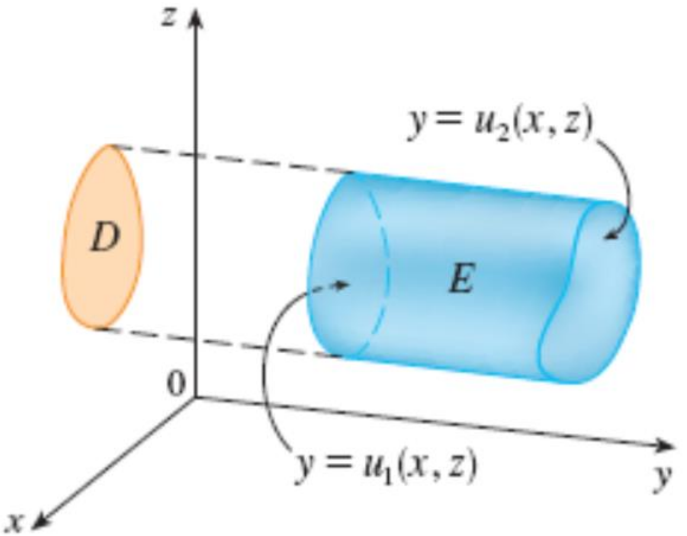
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right] dA$$

2. Triple Integrals Over General Bounded Regions

<u>Type 3 Regions (3-dim):</u>

A solid region E is a <u>type 3 region</u> if it is all points in space "to the right" of a plane region D in the xz-plane that are between the graphs of 2 surfaces (2-variable functions of x and z) $u_1(x,z)$ and $u_2(x,z)$.

$$E = \{ (x,y,z) \in \mathbb{R}^3 \,\big|\, (x,z) \in D \;\; and \;\; u_1(x,z) \le y \le u_2(x,z) \}$$



D will be a rectangle,
(2-dim) type 1 or type 2 region,
or a region we can easily describe
in polar coordinates

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA$$

2. Triple Integrals Over General Bounded Regions

Ex 2: Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

2. Triple Integrals Over General Bounded Regions

Ex 3: Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

2. Triple Integrals Over General Bounded Regions

Ex 4: Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^{y} f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x , then z , then y.

2. Triple Integrals Over General Bounded Regions

Ex 5: Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

2. Triple Integrals Over General Bounded Regions

Ex 6: Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.